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A two-dimensional inverse problem in imaging the thermal conductivity of a non-homogeneous medium

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Abstract

A two-dimensional inverse heat conduction problem is solved successfully by the conjugate gradient method (CGM) of minimization in imaging the unknown thermal conductivity of a non-homogeneous material. This technique can readily be applied to medical optical tomography problem. It is assumed that no prior information is available on the functional form of the unknown thermal conductivity in the present study, thus, it is classified as the function estimation in inverse calculation. The accuracy of the inverse analysis is examined by using simulated exact and inexact measurements obtained on the medium surface. The advantages of applying the CGM in the present inverse analysis lie in that the initial guesses of the unknown thermal conductivity can be chosen arbitrarily and the rate of convergence is fast. Results show that an excellent estimation on the thermal conductivity can be obtained within a couple of minutes CPU time at Pentium II-350 MHz PC. Finally the exact and estimated images of the thermal conductivity will be presented.

1. Introduction

A two-dimensional inverse heat conduction problem (IHCP) is examined in the present study by the conjugate gradient method to estimate the thermal conductivity of a non-homogeneous medium. In addition, temperature readings using infrared scanners taken at some appropriate locations and time on the medium surface are also considered available.

Numerous engineering and mathematical researchers have considered problems equivalent to estimating the thermal conductivity. For instance, Huang and Ozisik [1,2] used direct integration and Levenberg–Marquardt methods to estimate thermal conductivity and heat capacity simultaneously; Beck and Al-Araji [3] determined the constant thermal conductivity, heat capacity and contact conductance at one time; Terrola [4] used Davidon–Fletcher–Powell method to determine temperature-dependent thermal conductivity. All the above references belong to parameter estimations, i.e., the functional form for the unknown quantities should be assigned before the inverse calculations. However, when the thermal conductivity of a non-homogeneous or composite material is to be estimated, then from parameter estimation, it is difficult to achieve the goal, especially for multi-dimensional problems. Thus, function estimation with conjugate gradient method (CGM) [5] should be used in this inverse heat conduction problem to estimate the unknown thermal conductivity.

Recently, Huang and Yuan have developed an efficient function estimation algorithm based on the

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Nomenclature

J	functional defined by Eq. (2)
J'	gradient of functional defined by Eq. (11)
$k(x, y, t)$	unknown thermal conductivity
P	direction of descent defined by Eq. (3b)
$T(x, y, t)$	estimated dimensionless temperature
$\Delta T(x, y, t)$	sensitivity function defined by Eqs. (4a)–(4f)
$Y(x, y, t)$	measured temperature

Greek symbols

β	search step size
γ	conjugate coefficient
$\lambda(x, y, t)$	Lagrange multiplier defined by Eqs. (9a)–(9f)
$\delta(\cdot)$	Dirac delta function
ω	random number
ε	convergence criteria

Superscripts

$\hat{}$	estimated values
$\bar{}$	dimensional parameters
n	iteration index

Subscript

r	reference parameters

CGM in determining the thermal properties of the material in one-dimensional inverse problems. For example, Huang and Yuan [6] used the CGM in estimating temperature-dependent thermal conductivity. Huang and Yuan [7] used the same technique to estimate the temperature-dependent heat capacity per unit volume. Finally, Huang and Yuan [8] determined simultaneously the temperature-dependent thermal conductivity and heat capacity per unit volume.

The estimation of the thermal properties for the multi-dimensional problems is very limited in the literature. The purpose of the present study is to extend our previous algorithm to a two-dimensional IHCP to estimate the spatial and time-varying thermal conductivity in a non-homogeneous medium.

The CGM derives basis from the perturbation principle [5] and transforms the direct problem to the solution of two other related problems in the inverse analysis, namely, the sensitivity problem and the adjoint problem, which will be discussed in details in text.

Once this technique is established, it can also be applied to many applications in estimating the diffusion coefficients, such as the medical optical tomography problem [9,10], since the governing equations for those fields are very similar or even identical (for example in Ref. [10]) to the present study.

2. Direct problem

To illustrate the methodology of the present two-dimensional IHCP in determining unknown spatial and time varying thermal conductivity $k(x, y, t)$ in a non-homogeneous medium, we consider the following transient heat conduction problem.

A square plane Ω with side length $\bar{L}$ is initially at temperature $\bar{T}(\bar{x}, \bar{y}, 0) = \bar{T}_0$. For time $\bar{t} > 0$, the boundary surfaces along $\bar{x} = 0$ and $\bar{y} = 0$ are subjected to a prescribed constant heat flux $\bar{q}_1$ and $\bar{q}_3$, respectively, while along boundary surfaces $\bar{x} = \bar{L}$ and $\bar{y} = \bar{L}$, a constant heat flux $\bar{q}_2$ and $\bar{q}_4$ are taken away from the boundary by cooling, respectively. Fig. 1 shows the geometry and the coordinates for the two-dimensional physical problem considered here. The mathematical formulation of this transient heat conduction problem in dimensionless form is given as follows :

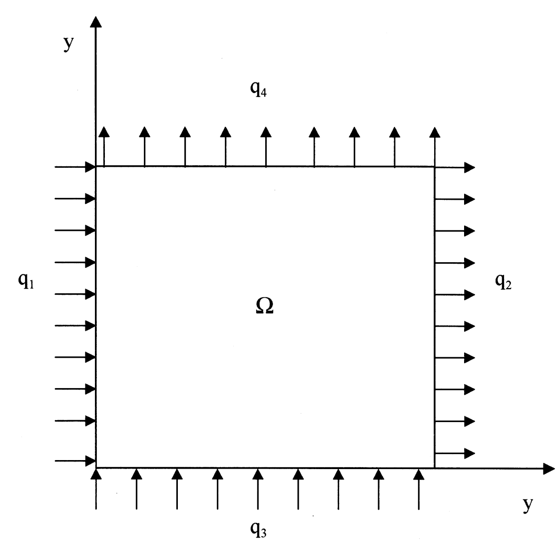

Fig. 1. Physical problem.

$$\frac{\partial}{\partial x}\left[k(x, y, t)\frac{\partial T(x, y, t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x, y, t)\frac{\partial T(x, y, t)}{\partial y}\right] = \frac{\partial T(x, y, t)}{\partial t} \quad \text{in } \Omega \tag{1a}$$

$$-k(x, y, t)\frac{\partial T(x, y, t)}{\partial x} = q_1 \quad \text{along } x = 0 \tag{1b}$$

$$-k(x, y, t)\frac{\partial T(x, y, t)}{\partial x} = q_3 \quad \text{along } x = L \tag{1c}$$

$$-k(x, y, t)\frac{\partial T(x, y, t)}{\partial y} = q_2 \quad \text{along } y = 0 \tag{1d}$$

$$-k(x, y, t)\frac{\partial T(x, y, t)}{\partial y} = q_4 \quad \text{along } y = L \tag{1e}$$

$$T(x, y, t) = 0 \quad \text{at } t = 0 \tag{1f}$$

where the following dimensionless quantities were defined

$$x = \frac{\bar{x}}{\bar{L}_r}, \quad y = \frac{\bar{y}}{\bar{L}_r}, \quad T = \frac{\bar{T} - \bar{T}_0}{\bar{T}_r}, \quad k = \frac{\bar{k}}{\bar{k}_r}, \quad q = \frac{\bar{L}}{\bar{k}_r \bar{T}_r}\bar{q},$$

$$t = \frac{\bar{k}_r}{\bar{\rho}\bar{C}_p \bar{L}_r^2}\bar{t}$$

where the superscript $^{-}$ denotes the dimensional quantities; $\bar{\rho}\bar{C}_p$ is the heat capacity per unit volume and $\bar{T}_r$ and $\bar{k}_r$ refer to the non-zero reference temperature and thermal conductivity, respectively. $\bar{L}_r$ represents the reference length.

The direct problem considered here is concerned with the determination of the medium temperatures when the thermal conductivity $k(x, y, t)$ and the initial and boundary conditions on the boundaries of Ω are known. The technique of alternating directional implicit (ADI) method is used to solve this direct problem.

3. Inverse problem

For the inverse problem, the thermal conductivity $k(x, y, t)$ is regarded as being unknown, but Eqs. (1) quantities in are known. In addition, temperature readings using infrared scanner taken at some appropriate grid locations and time on the medium surface are also considered available.

We assumed that the temperatures obtained from infrared scanner at the grid point are used to identify $k(x, y, t)$ in the inverse calculations. Let the temperature reading taken at these grid points over the time period t_f be denoted by $Y_{i,j}(X_i, Y_j, t) \equiv Y_{i,j}(t)$, $i = 1$ to I and $j = 1$ to I, where I represents the number of grid in x and y directions. We note that the measured temperature $Y_{i,j}(t)$ should contain measurement errors.

Then the inverse problem can be stated as follows: by utilizing the above-mentioned measured temperature data $Y_{i,j}(t)$, estimate the unknown $k(x, y, t)$ over the entire space and time domain.

Since all the measured temperatures are used to compute the entire unknown function for one period of time variation and no priori information is available on the functional form of $k(x, y, t)$, therefore, the method used here may be classified as the function estimation in the whole-domain [11] for the determination of the thermal conductivity in a non-homogeneous medium.

The solution of the present inverse problem is to be obtained in such a way that the following functional is minimized:

$$J\big[k(x, y, t)\big] = \int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\big[T_{i,j}(x_i, y_j, t) - Y_{i,j}(x_i, y_j, t)\big]^2 \,\mathrm{d}t \tag{2}$$

Here, $T_{i,j}$ are the estimated temperatures on the plane at the grid locations (x_i, y_j). These quantities are determined from the solution of the direct problem given previously by using an estimated $\hat{k}(x, y, t)$ for the exact $k(x, y, t)$. Here the superscript "^" denotes the estimated quantities.

4. Conjugate gradient method for minimization

The following iterative process based on the CGM [5] is now used for the estimation of $k(x, y, t)$ by minimizing the above functional $J[k(x, y, t)]$

$$\hat{k}^{n+1}(x, y, t) = \hat{k}^{n}(x, y, t) - \beta^n P^n(x, y, t) \quad \text{for } n = 0, 1, 2, \ldots \tag{3a}$$

where β^n is the search step size in going from iteration n to iteration $n+1$, and $P^n(x, y, t)$ is the direction of descent (i.e., search direction) given by

$$P^n(x, y, t) = J'^n(x, y, t) + \gamma^n P^{n-1}(x, y, t) \tag{3b}$$

which is a conjugation of the gradient direction

$J'^n(x, y, t)$ at iteration n and the direction of descent $P^{n-1}(x, y, t)$ at iteration $n-1$. The conjugate coefficient is determined from

$$\gamma^n = \frac{\int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} (J'^n)^2 \, dt\, dx\, dy}{\int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} (J'^{n-1})^2 \, dt\, dx\, dy} \quad \text{with } \gamma^0 = 0 \tag{3c}$$

We note that when $\gamma^n = 0$ for any n, in Eq. (3b), the direction of descent $P^n(x, y, t)$ becomes the gradient direction, i.e., the "Steepest descent" method is obtained. The convergence of the above iterative procedure in minimizing the functional J is guaranteed [12].

To perform the iterations according to Eqs. (3), we need to compute the step size β^n and the gradient of the functional $J'^n(x, y, t)$. In order to develop expressions for the determination of these two quantities, a "sensitivity problem" and an "adjoint problem" are constructed as described below.

5. Sensitivity problem and search step size

The sensitivity problem is obtained from the original direct problem defined by Eqs. (1) in the following manner. It is assumed that when $k(x, y, t)$ undergoes a variation $\Delta k(x, y, t)$, $T(x, y, t)$ is perturbed by $\Delta T(x, y, t)$. Then in the direct problem, replacing k by $k + \Delta k$ and T by $T + \Delta T$, subtracting from the resulting expressions the direct problem and neglecting the second-order terms, the following sensitivity problem for the sensitivity function ΔT is obtained:

$$\frac{\partial}{\partial x}\left[k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial y}\right] + \frac{\partial}{\partial x}\left[\Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial y}\right] = \frac{\partial \Delta T(x, y, t)}{\partial t} \quad \text{in } \Omega \tag{4a}$$

$$-k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial x} = \Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial x} \quad \text{along } x = 0 \tag{4b}$$

$$-k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial x} = \Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial x} \quad \text{along } x = L \tag{4c}$$

$$-k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial y} = \Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial y} \quad \text{along } y = 0 \tag{4d}$$

$$-k(x, y, t)\frac{\partial \Delta T(x, y, t)}{\partial y} = \Delta k(x, y, t)\frac{\partial T(x, y, t)}{\partial y} \quad \text{along } y = L \tag{4e}$$

$$\Delta T(x, y, t) = 0 \quad \text{at } t = 0 \tag{4f}$$

The technique of ADI method is used to solve this sensitivity problem.

The functional $J(\hat{k}^{n+1})$ for iteration $n+1$ is obtained by rewriting Eq. (2) as

$$J\left(\hat{k}^{n+1}\right) = \int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\left[T_{i,j}\left(\hat{k}^n - \beta^n P^n\right) - Y_{i,j}\right]^2 dt \tag{5a}$$

where we replaced $\hat{k}^{n+1}$ by the expression given by Eq. (3a). If temperature $T_{i,j}(\hat{k}^n - \beta^n P^n)$ is linearized by a Taylor expansion, Eq. (5a) takes the form

$$J\left(\hat{k}^{n+1}\right) = \int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\left[T_{i,j}\left(\hat{k}^n\right) - \beta^n \Delta T_{i,j}(P^n) - Y_{i,j}\right]^2 dt \tag{5b}$$

where $T_{i,j}(\hat{k}^n)$ is the solution of the direct problem by using estimate $\hat{k}^n(x, y, t)$ for exact $k(x, y, t)$ at the location (x_i, y_j). The sensitivity function $\Delta T_{i,j}(p^n)$ is taken as the solution of problem (4) at the grid positions (x_i, y_j) by letting $\Delta k = P^n$ [5]. The search step size β^n is determined by minimizing the functional given by Eq. (5b) with respect to β^n. The following expression results:

$$\beta^n = \frac{\int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\left(T_{i,j} - Y_{i,j}\right)\Delta T_{i,j}\, dt}{\int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\left(\Delta T_{i,j}\right)^2 dt} \tag{6}$$

6. Adjoint problem and gradient equation

To obtain the adjoint problem, Eq. (1a) is multiplied by the Lagrange multiplier (or adjoint function) $\lambda(x, y, t)$ and the resulting expression is integrated over the correspondent time and space domains. Then the result is added to the right-hand side of Eq. (2) to yield the following expression for the functional $J[k(x, y, t)]$:

$$J\big[k(x,y,t)\big] = \int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I}\big[T_{i,j} - Y_{i,j}\big]^2\,dt + \int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} \lambda(x,y,t) \times \left\{ \frac{\partial}{\partial x}\left[k(x,y,t)\frac{\partial T(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y,t)\frac{\partial T(x,y,t)}{\partial y}\right] - \frac{\partial T(x,y,t)}{\partial t}\right\} dt\,dx\,dy \tag{7}$$

The variation ΔJ is obtained by perturbing k by Δk and T by ΔT in Eq. (7), subtracting from the resulting expression the original equation (7) and neglecting the second-order terms. We thus deduce

$$\Delta J = \int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} \sum_{i=1}^{I}\sum_{j=1}^{I} 2(T-Y)\Delta T \delta(x-x_i)\delta(y - y_j)\,dt\,dx\,dy + \int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} \lambda \left\{ \frac{\partial}{\partial x}\left[k(x,y,t)\frac{\partial \Delta T(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y,t)\frac{\partial \Delta T(x,y,t)}{\partial y}\right] + \frac{\partial}{\partial x}\left[\Delta k(x,y,t)\frac{\partial T(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[\Delta k(x,y,t)\frac{\partial T(x,y,t)}{\partial y}\right] - \frac{\partial \Delta T(x,y,t)}{\partial t}\right\} dt\,dx\,dy \tag{8}$$

where $\delta(\cdot)$ is the Dirac delta function. In Eq. (8), the integral terms containing space or time derivative are integrated by parts; the initial and boundary conditions of the sensitivity problem given by Eqs. (4b)–(4d) are utilized. The vanishing of the integrands containing ΔT leads to the following adjoint problem for the determination of $\lambda(x, y, t)$:

$$\frac{\partial}{\partial x}\left[k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial y}\right] + \sum_{i=2}^{I-1}\sum_{j=2}^{I-1} 2(T - Y)\delta(x-x_i)\delta(y-y_j) + \frac{\partial \lambda(x,y,t)}{\partial t} = 0 \quad \text{in } \Omega \tag{9a}$$

$$-k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial x} = 2(T-Y) \quad \text{along } x = 0 \tag{9b}$$

$$k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial x} = 2(T-Y) \quad \text{along } x = L \tag{9c}$$

$$-k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial y} = 2(T-Y) \quad \text{along } y = 0 \tag{9d}$$

$$k(x,y,t)\frac{\partial \lambda(x,y,t)}{\partial y} = 2(T-Y) \quad \text{along } y = L \tag{9e}$$

$$\lambda(x,y,t) = 0 \quad \text{at } t = t_f \tag{9f}$$

The adjoint problem is different from the standard initial value problems in that the final time conditions at time $t = t_f$ is specified instead of the customary initial condition. However, this problem can be transformed to an initial value problem by the transformation of the time variables as $\tau = t_f - t$. Then the standard technique of ADI can be used to solve the adjoint problem.

Finally, the following integral term is left

$$\Delta J = \int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} -\left[\frac{\partial \lambda(x,y,t)}{\partial x}\frac{\partial T(x,y,t)}{\partial x} + \frac{\partial \lambda(x,y,t)}{\partial y}\frac{\partial T(x,y,t)}{\partial y}\right]\Delta k(x,y,t)\,dt\,dx\,dy \tag{10a}$$

From Ref. [5], we know that

$$\Delta J = \int_{x=0}^{L}\int_{y=0}^{L}\int_{t=0}^{t_f} J'(x,y,t)\Delta k(x,y,t)\,dt\,dx\,dy \tag{10b}$$

Then function $J'(x, y, t)$ is called a gradient of functional. A comparison of Eqs. (10a) and (10b) leads to the following expression for the gradient $J'(x, y, t)$ of the functional $J[k(x, y, t)]$:

$$J'(x, y, t) = -\left[\frac{\partial\lambda(x, y, t)}{\partial x}\frac{\partial T(x, y, t)}{\partial x} + \frac{\partial\lambda(x, y, t)}{\partial y}\frac{\partial T(x, y, t)}{\partial y}\right] \tag{11}$$

We note that $J'(x, y, t_f)$ is always equal to zero since $\lambda(x, y, t_f) = 0$, therefore, if the final time values of $k(x, y, t_f)$ cannot be predicted before the inverse calculation, the estimated values of $k(x, y, t)$ will deviate from exact values near final time condition [5]. This is the case in the present study! However, if we let $\lambda(x, y, t_f) = \lambda(x, y, t_f - \Delta t)$, where Δt denotes the time increment used in finite difference calculation, the singularity at $t = t_f$ can be avoided in the present study and a reliable inverse solution can be obtained [6–8].

7. Stopping criterion

If the problem contains no measurement errors, the traditional check condition is specified as

$$J[\hat{k}^{n+1}(x, y, t)] < \varepsilon \tag{12}$$

where ε is a small-specified number. However, the observed temperature data may contain measurement errors. Therefore, we do not expect the functional equation (2) to be equal to zero at the final iteration step. Following the experience of the authors [5–8], we use the discrepancy principle as the stopping criterion, i.e., we assume that the temperature residuals may be approximated by

$$T_{i,j} - Y_{i,j} \approx \sigma \tag{13}$$

where σ is the standard deviation of the measurements, which is assumed to be a constant. The above assumption was also made by Tikhonov [13] in order to find the optimal regularization parameter. Substituting Eq. (13) into Eq. (2), the following expression is obtained for ε:

$$\varepsilon = I^2\sigma^2 t_f \tag{14}$$

Then, the stopping criterion is given by Eq. (12) with ε determined from Eq. (14).

8. Computational procedure

The computational procedure for the solution of this inverse problem may be summarized as follows:

Suppose $\hat{k}^n(x, y, t)$ is available at iteration n.

Step 1. Solve the direct problem given by Eq. (1) for $T(x, y, t)$.

Step 2. Examine the stopping criterion given by Eq. (12) with ε given by Eq. (14). Continue if not satisfied.

Step 3. Solve the adjoint problem given by Eq. (9) for $\lambda(x, y, t)$.

Step 4. Compute the gradient of the functional $J'(x, y, t)$ from Eq. (11).

Step 5. Compute the conjugate coefficient γ^n and direction of descent P^n from Eqs. (3c) and (3b), respectively.

Step 6. Set $\Delta k(x, y, t) = P^n(x, y, t)$, and solve the sensitivity problem given by Eq. (4) for $\Delta T(x, y, t)$.

Step 7. Compute the search step size β^n from Eq. (6).

Step 8. Compute the new estimation for $\hat{k}^{n+1}(x, y, t)$ from Eq. (3a) and return to step 1.

9. Results and discussions

To demonstrate the validity of the present CGM in predicting $k(x, y, t)$ for a two-dimensional non-homogeneous material from the knowledge of transient temperature recordings on the medium surface, we consider two specific examples where a drastic change of the thermal conductivity is considered.

The objective of this article is to show the accuracy of the present approach in estimating $k(x, y, t)$ with no prior information on the functional form of the unknown quantities, which is the so-called function estimation.

In order to compare the results for situations involving random measurement errors, we assume normally distributed uncorrelated errors with zero mean and constant standard deviation. The simulated inexact measurement data $\boldsymbol{Y}$ can be expressed as

$$Y = Y_{\text{exact}} + \omega\sigma \tag{15}$$

where Y_{exact} is the solution of the direct problem with an exact $k(x, y, t)$; σ is the standard deviation of the measurements; and ω is a random variable that generated by subroutine DRNNOR of the IMSL [14] and will be within −2.576 to 2.576 for a 99% confidence bounds.

One of the advantages of using the CGM is that the initial assumptions of the unknown quantities can be chosen arbitrarily. In all the test cases considered here, the initial assumptions of $k(x, y, t)$ used to begin the iteration are taken as $\hat{k}(x, y, t)_{\text{initial}} = 0.1$. Besides, the side length of domain Ω is taken as $L = 12$; the space increment is taken as $\Delta x = \Delta y = 0.5$ in the finite difference calculations, therefore $I = 25$. The total measurement time is chosen as $t_f = 10$ and time increment is chosen as $\Delta t = 0.5$. The boundary heat fluxes are taken as $q_1 = 100$, $q_2 = 60$, $q_3 = 100$ and $q_4 = 60$; and measurement time step Dt is taken to be the same as Δt; therefore, total of 12,500 discrete number of $k(x, y, t)$ are to be estimated simultaneously in the present inverse calculations.

We now present two numerical experiments in determining $k(x, y, t)$ or $k(i, j, m)$ by the inverse analysis. Here i and j represent the grid index of the space while m denotes the grid index of time.

9.1. Numerical test case 1

A three-region non-homogeneous thermal conductivity $k(i, j, m)$ is assumed to vary with the position and time in the domain Ω as stated below:

1. For time $0 < t \leq 3$, the exact distributions for thermal conductivity $k(i, j, m)$ are shown in Fig. 2, where the values of k in regions I, II and III are assumed to be 6, 8 and 10, respectively.
2. For time $3 < t \leq 6.5$, the exact distributions for thermal conductivity $k(i, j, m)$ are shown in Fig. 3, where the values of k in regions I, II and III are assumed to be 10, 9 and 12, respectively.
3. For time $6.5 < t \leq 10$, the exact distributions for thermal conductivity $k(i, j, m)$ are shown in Fig. 4, where the values of k in region I, II and III are assumed to be 17, 16 and 14, respectively.

The inverse analysis is first performed by assuming exact measurements, $\sigma = 0$, i.e., no measurement errors. By setting the number of iteration equals to 70, after about 5 min and 15 s CPU time at Pentium II-350 MHz PC, the estimated thermal conductivity $k(i, j, m)$ can be obtained.

The estimated values of $k(i, j, m)$ are in good agreement with the exact values since the relative average errors for the estimated temperatures and thermal conductivity are ERR1 = 0.26% and ERR2 = 3.5%, respectively, where the definitions for ERR1 and ERR2 are shown below:

$$\mathrm{ERR1}\ (\%) = \sum_{i=1}^{25}\sum_{j=1}^{25}\sum_{m=1}^{20}\left|\frac{T(i,j,m) - Y(i,j,m)}{Y(i,j,m)}\right| \div (25 \times 25 \times 20) \times 100\% \quad (16)$$

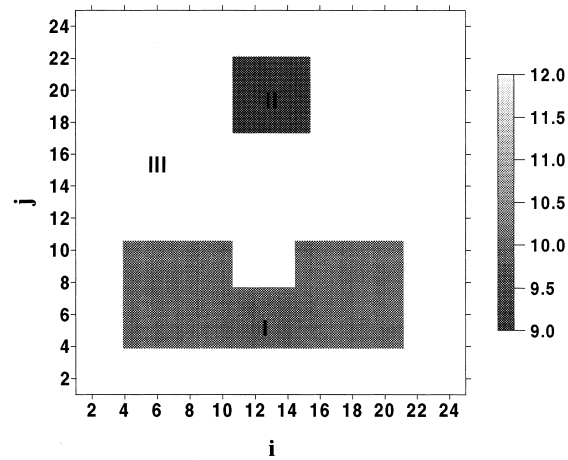

Fig. 3. The exact image of thermal conductivity $k(i, j, 6)$ for test case 1 at $t = 6$.

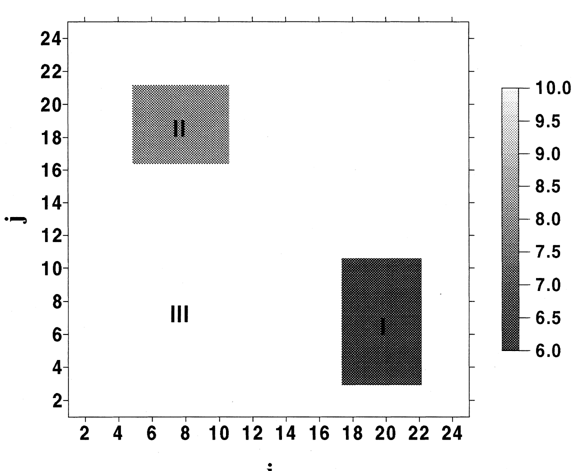

Fig. 2. The exact image of thermal conductivity $k(i, j, 2.5)$ for test case 1 at $t = 2.5$.

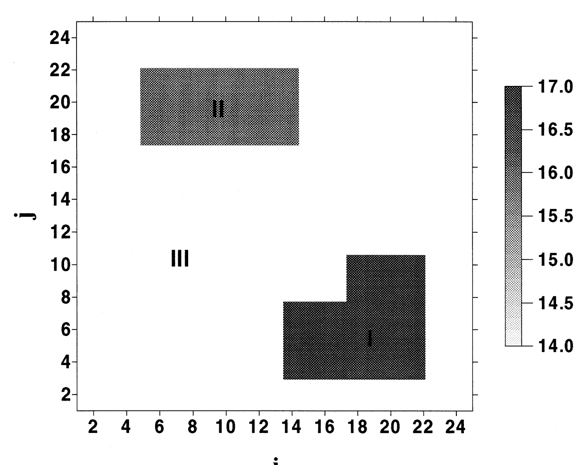

Fig. 4. The exact image of thermal conductivity $k(i, j, 9.5)$ for test case 1 at $t = 9.5$.

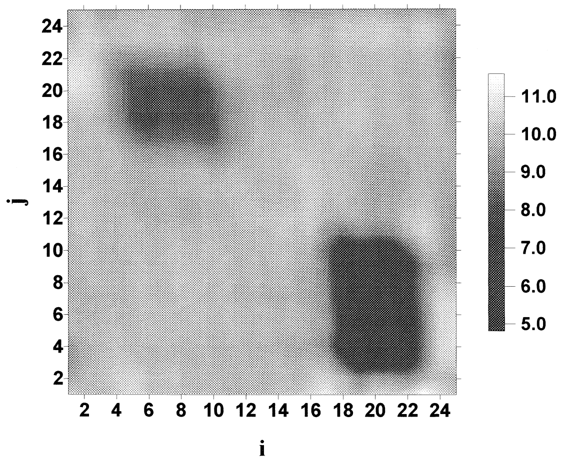

Fig. 5. The estimated image of thermal conductivity $k(i, j, 2.5)$ for test case 1 at $t = 2.5$ with $\sigma = 0.0$.

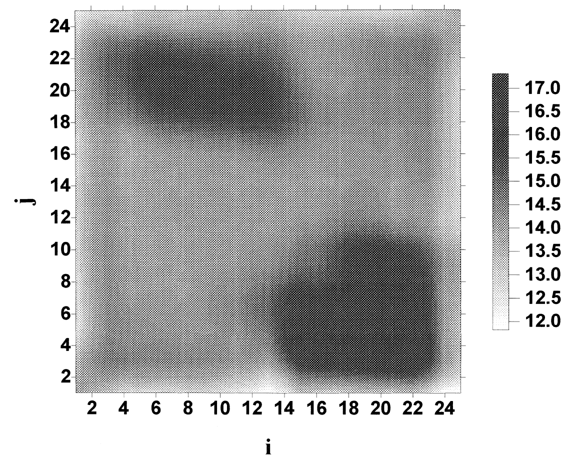

Fig. 7. The estimated image of thermal conductivity $k(i, j, 9.5)$ for test case 1 at $t = 9.5$ with $\sigma = 0.0$.

$$\text{ERR2}\ (\%) = \sum_{i=1}^{25}\sum_{j=1}^{25}\sum_{m=1}^{20}\left|\frac{\hat{k}(i,j,m) - k_{\text{exact}}(i,j,m)}{k_{\text{exact}}(i,j,m)}\right| \div (25 \times 25 \times 20) \times 100\% \tag{17}$$

The estimated images for thermal conductivity at time $t = 2.5$, 6 and 9.5 are shown in Figs. 5–7, respectively. From these figures we learn that the tomography technique by applying the CGM in estimating the unknown thermal conductivity in a non-homogeneous medium is now complete.

Next, let us discuss the influence of the measurement errors on the inverse solutions. First, the measurement error for the infrared scanner is taken as $\sigma = 0.35$ (about 1% of the average measured temperature), then error is increased to $\sigma = 1.05$ (about 3% of the average measured temperature). The inverse calculations are then performed and the stopping criteria based on the discrepancy principle is applied.

For the case when $\sigma = 0.35$, after 50 iterations (CPU time used at Pentium II-350 MHz PC is about 3 min and 45 s), the inverse solutions are converged. The relative average errors for the estimated temperatures and thermal conductivity are calculated as ERR1 = 4.3% and ERR2 = 5.4%, respectively, and the estimated images at time t = 2.5, 6 and 9.5 are shown in Figs. 8–10, respectively.

When $\sigma = 0.35$, after 13 iterations (CPU time used at Pentium II-350 MHz PC is about 48 s), the sol-

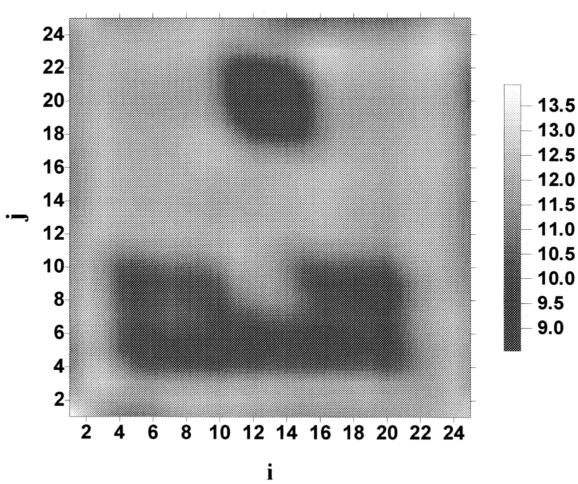

Fig. 6. The estimated image of thermal conductivity $k(i, j, 6)$ for test case 1 at $t = 9.5$ with $\sigma = 0.0$.

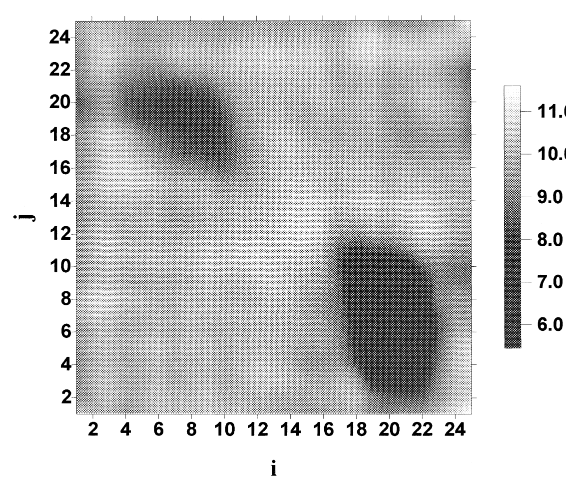

Fig. 8. The estimated image of thermal conductivity $k(i, j, 2.5)$ for test case 1 at $t = 2.5$ with $\sigma = 0.35$.

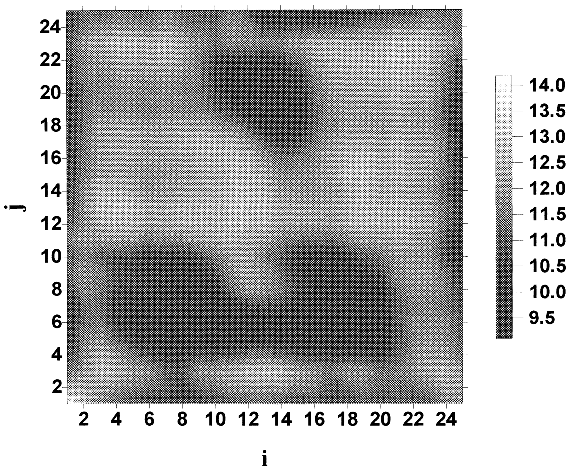

Fig. 9. The estimated image of thermal conductivity $k(i, j, 6)$ for test case 1 at $t = 6$ with $\sigma = 0.35$.

utions for the estimated thermal conductivity are converged. The relative average errors for the estimated temperatures and thermal conductivity are calculated as ERR1 = 11.6% and ERR2 = 10.1%, respectively.

9.2. Numerical test case 2

In the second test case, we assumed that the functions of thermal conductivity are in a more complicated form. They can be specified as follows:

1. For time $0 < t \leq 5$, the exact distributions for thermal conductivity $k(i, j, m)$ in regions I, II and III are given as:

I:

$$k(i, j, m) = 6 + 0.1 \times (i + j + m) + \sin(2 \times \pi \times m/20)$$

II:

$$k(i, j, m) = 9 + 0.1 \times (i + j + m) + \sin(2 \times \pi \times m/20)$$

III:

$$k(i, j, m) = 12 + 0.1 \times (i + j + m) + \sin(2 \times \pi \times m/20)$$

2. For time $5 < t \leq 10$, t the exact distributions for thermal conductivity $k(i, j, m)$ in regions I and II are given as:

I:

$$k(i, j, m) = 11 + 0.1 \times (i + j + m) + 5 \times \sin(2 \times \pi \times m/20)$$

II:

$$k(i, j, m) = 14 + 0.1 \times (i + j + m) + 5 \times \sin(2 \times \pi \times m/20)$$

When assuming exact measurements, $\sigma = 0$, (i.e., no measurement errors), and setting the number of iteration equals to 50. After about 4 min and 25 s CPU time at Pentium II-350 MHz PC, the inverse solutions in estimating the thermal conductivity $k(i, j, m)$ can be

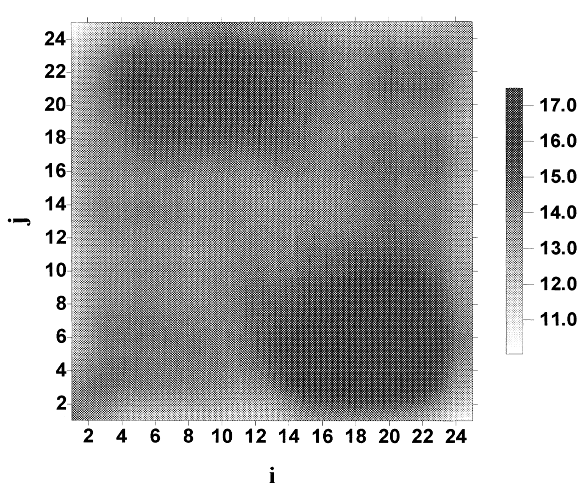

Fig. 10. The estimated image of thermal conductivity $k(i, j, 9.5)$ for test case 1 at $t = 9.5$ with $\sigma = 0.35$.

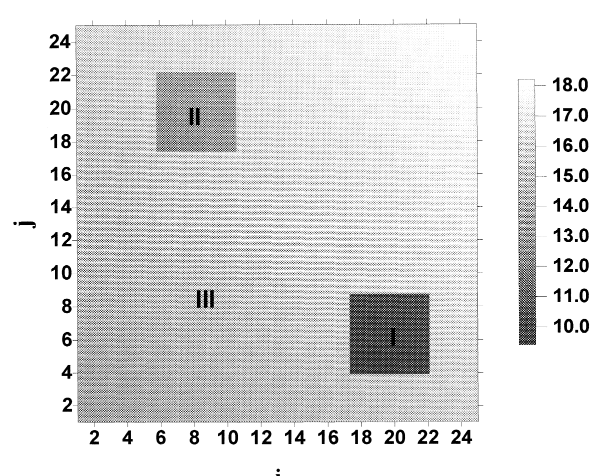

Fig. 11. The exact image of thermal conductivity $k(i, j, 4.5)$ for test case 2 at $t = 4.5$.

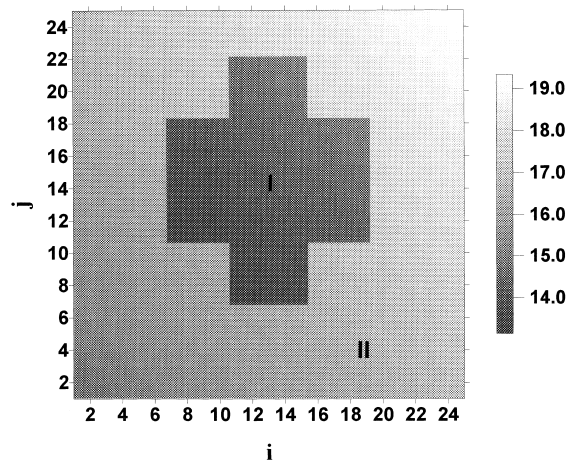

Fig. 12. The exact image of thermal conductivity $k(i, j, 9.5)$ for test case 2 at $t = 9.5$.

obtained. The relative average errors for the estimated temperatures and thermal conductivity are ERR1 = 1.6% and ERR2 = 4.7%, respectively. This implies that the estimated values of $k(i, j, m)$ are also in good agreement with the exact values in this test case.

The exact images for the thermal conductivity at time $t = 4.5$ and 9.5 are shown in Figs. 11 and 12, respectively, while the estimated images for the same time are shown in Figs. 13 and 14, respectively. From these figures we learn that the CGM can also be applied in this thermal conductivity tomography estimation.

Next, similar to the previous procedure, we will discuss the influence of the measurement errors on the inverse solutions. First, the measurement error for the infrared scanner is taken as $\sigma = 0.31$ (about 1% of the average measured temperature), then error is increased to $\sigma = 1.02$ (about 3% of the average measured temperature). The inverse calculations are then performed and the stopping criteria based on the discrepancy principle is applied.

For the case when $\sigma = 0.31$, after 41 iterations (about 3 min and 37 s CPU time is used), the inverse solutions are converged. The relative average errors for the estimated temperatures and thermal conductivity are calculated as ERR1 = 3.7% and ERR2 = 6.1%, respectively.

When $\sigma = 1.02$, after 16 iterations (about 1 min and 25 s CPU time is used), the solutions for the estimated thermal conductivity are converged. The relative average errors for the estimated temperatures and thermal conductivity are calculated as ERR1 = 12.5% and ERR2 = 11.0%, respectively.

By using 1% and 3% measurement errors in these two test cases, one could estimate the thermal conductivity with average relative error with the order of about 5% and 10%, respectively. This represents that the measurement errors did not amplify the errors of estimated thermal conductivity and therefore the present technique provides a confidence estimation in determining 12,500 discrete number, of $k(i, j, m)$ simulataneously.

From the above numerical test cases 1 and 2, we concluded that the CGM can be applied successfully in the function estimation for predicting the non-homogeneous thermal conductivity with very fast speed of convergence!

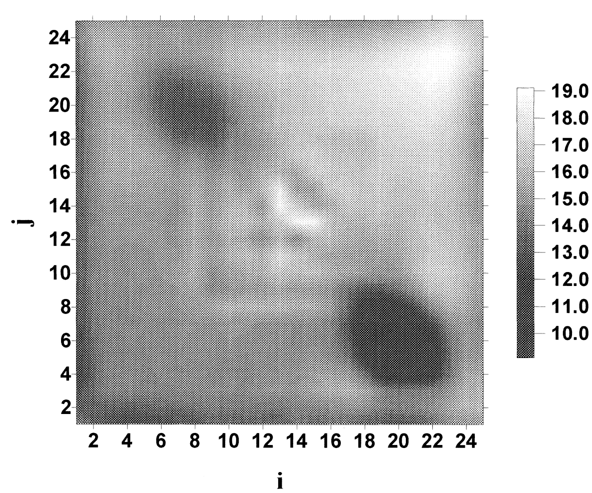

Fig. 13. The estimated image of thermal conductivity $k(i, j, 4.5)$ for test case 2 at $t = 4.5$ with $\sigma = 0.0$.

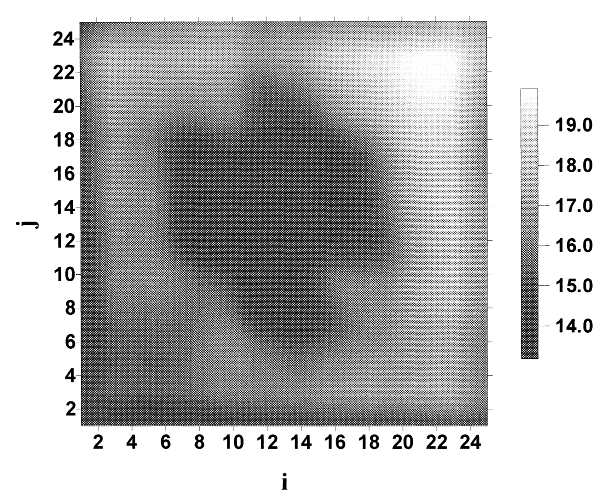

Fig. 14. The estimated image of thermal conductivity $k(i, j, 9.5)$ for test case 2 at $t = 9.5$ with $\sigma = 0.0$.

10. Conclusions

The CGM with adjoint equation was successfuly applied for the solution of the inverse problem to determine the non-homogeneous thermal conductivity. Several test cases involving different functional forms of non-homogeneous thermal conductivity and measurement errors were considered. The results show that the CGM does not require a priori information for the functional form of the unknown quantities and needs very short CPU time at Pentium II-350 MHz PC to perform the inverse calculations.

Acknowledgements

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